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Trace anomalies in a two-dimensional de Sitter metric and black-body radiation

Daksh Lohiya

Department of Physics and Astrophysics, University of Delhi, Delhi-110007, India

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Abstract. We investigate the possibility of a Hawking-type effect due to the cosmological event horizon. We consider the reduced two-dimensional de Sitter metric. There appears to be radiation from the horizon towards the origin at temperature T defined by

$$kT = \sqrt{\Lambda'}/2\pi$$

where $3\Lambda'$ is the cosmological constant. The observer dependence of this radiation is discussed.

1. Introduction

It is generally believed that the presence of an event horizon will induce creation of particles. In the case of a black hole, the positive energy flux radiated to infinity can be pictured as follows (Hawking 1975). The event horizon separates two regions such that the Killing vector representing time translation is time-like in one region and space-like in the other. In the former ('physical') region, virtual pairs of particles are created. The positive energy particle contributes to the energy flux at infinity and the negative energy particle tunnels through the horizon—decreasing the mass of the black hole in consistency with the energy flux at infinity.

We carry over these arguments to the case of the cosmological event horizon that arises in the case where the stress–energy tensor of general theory of relativity is

$$R^{\mu\nu} - \tfrac{1}{2}g^{\mu\nu}R + \Lambda g^{\mu\nu} = \kappa T^{\mu\nu} \tag{1}$$

with a non-zero Λ. For an empty space–time

$$R^{\mu\nu} = \Lambda g^{\mu\nu} \tag{2}$$

has, as its solution, the metric

$$ds^2 = -(1 - \tfrac{1}{3}\Lambda r^2)\, dt^2 + (1 - \tfrac{1}{3}\Lambda r^2)^{-1}\, dr^2 + r^2\, d\Omega^2 \tag{3}$$

where the coordinate r is understood such that the area of a sphere of radius r is $4\pi r^2$.

Near the event horizon ($r = \sqrt{(3/\Lambda)} \equiv 1/\sqrt{\Lambda'}$) virtual pairs may again be produced, the positive energy particle being radiated towards the origin (the region of the time-like Killing vector) and the negative energy particle 'going over to the other side'. We shall restrict ourselves to the two-dimensional analogue

$$ds^2 = -(1 - \tfrac{1}{3}\Lambda r^2)\, dt^2 + (1 - \tfrac{1}{3}\Lambda r^2)^{-1}\, dr^2 \tag{4}$$

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(We are still working on the complete four-dimensional case.) This metric may be understood to be a model that may be of help in the future complete four-dimensional model. Of course equation (1) does not imply equation (2) in the two-dimensional case, neither does equation (2) imply equation (4) in the two-dimensional case. However, direct evaluation of the curvature scalar for the metric (equation (4)) yields $R = 2\Lambda'$. We are just on a constant (θ, ϕ) surface of the de Sitter metric. We may choose to start with equation (4). In our model the metric is given by equation (4) only for $r \geqslant 0$. $r < 0$ is not defined for the complete four-dimensional de Sitter metric. Formally we consider 'reflection' at $r = 0$. We use Misner–Thorne–Wheeler (Misner *et al* 1973) sign conventions and shall closely follow the Christensen and Fulling (1977) treatment for the black hole.

In § 2 we derive the 'Hawking effect' (black-body effects) we are looking for. The relevant mathematical details have been outlined in the appendix.

2. The two-dimensional cosmological model

Christensen and Fulling have shown that the results of calculation for a two-dimensional Hawking problem can be reproduced by general physical arguments and by the knowledge of the trace anomaly of the two-dimensional scalar field. It is known (Davies *et al* 1976) that the trace anomaly coefficient is $T_\alpha{}^\alpha = R/24$. As it is known that: (i) $T_\mu{}^\nu$ conservation, (ii) zero trace, and (iii) particle production, are incompatible, we therefore consider the most general stress tensor that satisfies all the above conditions except the tracelessness and that has finite components with respect to a local orthonormal frame on the future cosmological horizon.

The most general solution of the conservation equation

$$\nabla_\nu T_\mu{}^\nu = 0 \tag{5}$$

for $T_\mu{}^\nu$ independent of time, for the cosmological background equation (4), may now be found. The conservation equations are

$$\begin{aligned} &\partial_r[(1-\Lambda' r^2)T_r^r] = -r\Lambda' T_\alpha{}^\alpha \\ &\partial_r T_t^r = 0. \end{aligned} \tag{6}$$

Defining

$$H(r) = \Lambda' \int_r^{(\Lambda')^{-1/2}} r' T_\alpha{}^\alpha(r')\,\mathrm{d}r'$$

and defining 'tortoise' coordinate r^* by $\mathrm{d}r/\mathrm{d}r^* = 1 - \Lambda r^2$ as in the appendix; the general solution to equation (6) is, in (t, r^*) *coordinates,*

$$T_\mu{}^\nu = T_\mu{}^{\nu(1)} + T_\mu{}^{\nu(2)} + T_\mu{}^{\nu(3)} \tag{7}$$

with

$$T_\mu{}^{\nu(1)} \equiv \begin{pmatrix} -(1-\Lambda' r^2)^{-1}H(r) + T_\alpha{}^\alpha(r) & 0 \\ 0 & (1-\Lambda' r^2)^{-1}H(r) \end{pmatrix}$$

$$T_\mu{}^{\nu(2)} \equiv K\Lambda'(1-\Lambda' r^2)^{-1}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}; \qquad T_\mu{}^{\nu(3)} \equiv Q\Lambda'(1-\Lambda' r^2)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{8}$$

where we have used $T_{r^*}{}^{r^*} = T_r{}^r$; $T_{r^*}{}^t = -T_t{}^{r^*}$ and $T_t{}^{r^*} = (1-\Lambda' r^2)^{-1} T_t{}^r$. The signs have been chosen to have $T_r{}^t$ positive for matter moving in the positive r direction. It can be shown by arguments similar to Christensen and Fulling (1977) that the stress tensor for black-body radiation in a two-dimensional flat space moving in the negative r direction is

$$T_\mu{}^{\nu(\mathrm{rad})} = \frac{\pi (kT)^2}{12} \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} \tag{9}$$

where T is the equilibrium temperature. Assigning a temperature to the radiation from the event horizon and comparing with (7) as $r \to 0$ gives

$$|\mathrm{flux}| = |K\Lambda'| = \tfrac{1}{12}\pi (kT)^2. \tag{10}$$

However, $T_\mu{}^{\nu(2)}$ does not represent the complete stress tensor as we can easily verify: $T_{uu}{}^{(2)} = -K\Lambda'$ and $T_{vv}{}^{(2)} = 0$. Thus with $K > 0$ it represents negative energy flowing into the event horizon. To recover the form of $T_\mu{}^{\nu(\mathrm{rad})}$ as $r \to 0$, we may take $Q = 2K$. This however, would imply an infinite intensity at the horizon. Before we go forward, it is necessary to define our vacuum state. If we choose our vacuum state that is invariant under all the Killing fields of the de Sitter metric then we would find $T_{\mu\nu}$ to be simply proportional to $Rg_{\mu\nu}$ ($K = 0$). In the path integral formulation of particle creation in cosmology, Gibbons and Hawking (1977) introduce asymmetric considerations, namely in equation (4.20) of their paper, where they assume there to be no particles present on the 'space-like surface' in the distant past. Thus the only contribution to the 'amplitude' comes from the surface in the future. When the future and past horizons are chosen to be in 'equilibrium' then no radiation occurs. However, to consider particle creation by the universe, one is interested only in those particles which were not present at the infinite past (Hawking, private communication). The process by which particle creation can be motivated is by the 'propogation' of the particle amplitude from a point near the de Sitter horizon to a surface outside the future horizon and the propagation from the original point towards the observer near the origin (Gibbons and Hawking 1977). Thus the vacuum we are to choose must give a regular stress tensor on the future horizon. In this (Unruh) vacuum (Unruh 1976) the future and past horizons are not in equilibrium and thus no state invariant under the isometries (Fulling 1977) can yield a non-singular stress tensor. We thus choose the state that has $T_{\mu\nu}$ finite on the future horizon and may be singular on the past. (We have shown (see appendix) that $T_{\mu\nu}$ as measured in local frames on the future horizon shall be finite if $T_t{}^t + T_{r^*}{}^{r^*}$ and T_{uu} are finite and

$$\lim_{r \to (\Lambda')^{-1/2}} \left(\frac{1 - \sqrt{\Lambda'} r}{1 + \sqrt{\Lambda'} r} \right)^{-2} |T_{vv}| < \infty. \tag{11}$$

This condition is violated if $Q = 2K$.)

The resolution of this difficulty is to assign $T_\mu{}^\nu$ a trace. With $Q = 0$ the total energy density ($= T_t{}^t$) approaches

$$-T_t{}^t \underset{r \to 0}{\to} H(0) - T_\alpha{}^\alpha(0) - K\Lambda'. \tag{12}$$

As the incoming radiation is in a thermal state, the energy density should be greater than the magnitude of the flux $K\Lambda'$. Therefore $T_\alpha{}^\alpha(r)$ cannot be identically zero.

Using

$$T_\alpha{}^\alpha(r) = R/24\pi = \Lambda'/12\pi \tag{13}$$

and using the case for massless particles, for which density and flux are equal, i.e.

$$H(0) - T_\alpha{}^\alpha(0) - K\Lambda' = K\Lambda' \tag{14}$$

we get,

$$\text{flux} = -K\Lambda' = \Lambda'/48\pi.$$

Thus from equation (9), we deduce that an observer near the origin should receive a black-body radiation corresponding to a temperature given by

$$kT = \sqrt{\Lambda'}/2\pi. \tag{15}$$

It is known (see for example Rindler 1969) that the complete de Sitter space can be mapped onto a five-dimensional pseudosphere. It follows that any point on the pseudosphere can be transformed onto the origin $r = t = 0$. Our (r, t) space covers only a part of this pseudosphere. However, we show in a manner exactly similar to Rindler that this too can be mapped onto a three-dimensional pseudosphere (a hyperboloid). Any point on this hyperboloid can be transformed to the origin of the (r, t) space.

Proof. Under the transformations (Rindler 1969)

$$\begin{pmatrix} W \\ T \end{pmatrix} = a\left(1 - \frac{r^2}{a^2}\right)^{1/2} \begin{pmatrix} \cosh(t/a) \\ \sinh(t/a) \end{pmatrix} \qquad r < a \equiv \left(\frac{3}{\Lambda}\right)^{1/2}$$

and

$$\begin{pmatrix} W \\ T \end{pmatrix} = a\left(\frac{r^2}{a^2} - 1\right)^{1/2} \begin{pmatrix} \sinh(t/a) \\ \cosh(t/a) \end{pmatrix} \qquad r > a \equiv \left(\frac{3}{\Lambda}\right)^{1/2}$$

a point (r, t) transforms into a point satisfying

$$W^2 + r^2 - T^2 = a^2. \tag{16}$$

Now (W, r, T) is space-like (from (16)), therefore, as a displacement along the surface of the pseudosphere (defined by (16)) is perpendicular to (W, r, t), the surface should contain the remaining one time-like and one space-like dimension—giving the right signature for an (r, t) space–time.

Consider a point $P_0 = P_0(W_0, r_0, t_0)$ on this pseudosphere. We can perform a two-rotation in r and W to give $R_0 = 0$ and then a two-rotation (a Lorentz transformation) ensuring $T_0 = 0$. Now as $r = t = 0$ implies $W = a$ which is the transformed point P_0, therefore the point P_0 is a map of $r = t = 0$. The transformed radial coordinate r' also has an event horizon at $r' = (\Lambda/3)^{-1/2}$. It follows, therefore, that every point in this model can choose a vacuum to interpret the incoming radiation as being of the temperature given by equation (15) and as being centred on it. The above mapping does not refer to an isometry. The radiation is same from the two perspectives (r and r') only if we consider a different quantum state in each case, constructed (by the procedure outlined in our paper) relative to the coordinate system associated with that case. Thus all observers moving on time-like geodesics will see the radiation at the same temperature even though they may be moving with respect to each other. Thus they do not observe the same particles (Gibbons and Hawking 1977, Hăjiček 1977).

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Appendix

The 'tortoise' coordinate for the metric

$$ds^2 = -(1-\Lambda' r^2)\,dt^2 + (1-\Lambda' r^2)^{-1}\,dr^2$$

may be defined, as done for the black hole by Misner *et al* (1973) by

$$\frac{dr}{dr^*} = 1-\Lambda' r^2 \qquad \Rightarrow \qquad r^* = \frac{1}{2\sqrt{\Lambda'}} \ln\left(\frac{1+\sqrt{\Lambda'}r}{1-\sqrt{\Lambda'}r}\right).$$

We define ingoing and outgoing coordinates as

$$v = t + r^*; \qquad u = t - r^*.$$

The coordinates appropriate for incoming waves through the horizon are the (v, r) coordinates. The metric takes the form

$$ds^2 = -(1-\Lambda' r^2)\,dv^2 + 2\,dv\,dr$$

for outgoing waves, the appropriate coordinates are (u, r),

$$ds^2 = -(1-\Lambda' r^2)\,du^2 - 2\,du\,dr.$$

In the (u, v) coordinate system, the metric takes the form

$$ds^2 = -(1-\Lambda' r^2)\,du\,dv$$

Thus the (u, v) system is not well behaved at $r = (\Lambda')^{-1/2}$. We can, however, define a 'Kruskal-type' coordinate system

$$U \equiv -e^{u\sqrt{\Lambda'}} = -e^{t\sqrt{\Lambda'}}\left(\frac{1+\sqrt{\Lambda'}r}{1-\sqrt{\Lambda'}r}\right)^{1/2}$$

and

$$V \equiv e^{-v\sqrt{\Lambda'}} = e^{-t\sqrt{\Lambda'}}\left(\frac{1+\sqrt{\Lambda'}r}{1-\sqrt{\Lambda'}r}\right)^{1/2}.$$

The metric becomes

$$ds^2 = \frac{(1+\sqrt{\Lambda'}r)^2}{\Lambda'}\,dU\,dV.$$

This does not have any offending behaviour at $r=(\Lambda')^{-1/2}$. A tensor is finite on the horizon, if and only if it is finite as measured in a regular coordinate system such as the (U, V) system just defined.

From

$$UV=\frac{1-\sqrt{\Lambda'}r}{1+\sqrt{\Lambda'}r}$$

and the definition of the (U, V) coordinates, we get

$$T_{UU}=\frac{U^{-2}}{\Lambda'}T_{uu}$$

$$T_{UV}=\frac{(UV)^{-1}}{\Lambda'}T_{uv}=\frac{1}{\Lambda'}\left(\frac{1-\sqrt{\Lambda'}r}{1+\sqrt{\Lambda'}r}\right)^{-1}T_{uv}$$

$$T_{VV}=\frac{V^{-2}}{\Lambda'}T_{vv}=\frac{U^{2}}{\Lambda'}\left(\frac{1-\sqrt{\Lambda'}r}{1+\sqrt{\Lambda'}r}\right)^{-2}T_{vv}.$$

Therefore $T_{\mu\nu}$ is 'physically finite' on the future horizon ($V=0$) if, as $r\to(\Lambda')^{-1/2}$,

(*a*) $|T_{uu}|$ is finite;

(*b*) $\left(\frac{1-\sqrt{\Lambda'}r}{1+\sqrt{\Lambda'}r}\right)^{-1}|T_{uv}|$

$$=\frac{(1+\sqrt{\Lambda'}r)^2}{4}|T_t{}^t+T_{r^*}{}^{r^*}| \text{ is finite, i.e. } |T_t{}^t+T_{r^*}{}^{r^*}| \text{ is finite;}$$

(*c*) $\left(\frac{1-\sqrt{\Lambda'}r}{1+\sqrt{\Lambda'}r}\right)^{-2}|T_{vv}|$ is finite.

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